REINFORCED CONCRETE COLUMNS IN BIAXIAL BENDING

ENERCALC, INC.

Introduction

This program provides analysis and design of arbitrarily shaped reinforced concrete columns loaded with axial loads and uni-axial or bi-axial bending moments. The user may compute the capacity of a pre-defined section or determine a design for pre-defined combinations of axial load and uni-axial or bi-axial bending moments. The applied loads can be either compressive or tensile axial loads either working in conjunction with or without uni-axial or bi-axial bending moments. Users may enter either service or ultimate load values for the axial loads and bending moments. Where service loads are selected the program will scale user inputs by the appropriate ACI-318 load factors and compare the pre-defined cross-section capacity with each load combination.

The computational method used within this module is based on the transformation of the double equilibrium integrals in the compression area to line integrals around the region's perimeter by employing Green's Theorem. Provided the stress-strain relationship of the concrete model can be expressed in a line integral form, the method yields an exact mathematical solution. In the case of the ACI-318 the stress-strain relationship of the concrete model is expressed in terms of a constant variable and therefore can be utilized in Green's theorem.

Assumptions

The commonly accepted assumptions and limitations used in reinforced concrete design are stated below:

1. Bernoulli's assumption that plane sections remain plane before and after bending is valid.
2. The strain in the concrete and the reinforcement is directly proportional to the distance from the neutral axis.
3. Effects of creep and shrinkage can be ignored.
4. Tensile strength of concrete is neglected.
5. Member does not buckle before the ultimate load is attained.
6. Column ties per ACI-318 are provided.
7. The Whitney uniform stress block is used. The maximum uniform rectangular stress is 0.85 f_c' and the depth of the stress block $a = \beta_1 c$. The value of β_1 is interdependent upon the concrete compressive strength as defined in ACI-318:

   $0.65 \leq \beta_1 = 1.05 - 0.05 f_c' \leq 0.85$ where f_c' is in ksi
   $0.65 \leq \beta_1 = 1.0643 - 0.007143 f_c' \leq 0.85$ where f_c' is in MPa

8. The maximum strain limit in the concrete is 0.0030 in./in. per the ACI-318 and Whitney stress block.
9. The modulus of elasticity of concrete is computed as follows:

   $E_c = 33\, \omega^{1.5} (f_c')^{1/2}$ where E_c and fc' are in ksi and ω is in pcf
   $E_c = 0.043\, \omega^{1.5} (f_c')^{1/2}$ where E_c and fc' are in MPa and ω is in kg/m^3

10. Moments may be computed about either the geometric centroid (G.C.) of the cross-section (neglecting the steel reinforcement) or the plastic centroid (P.C.) of the cross-section. The plastic centroid coordinate system can be determined when the section is in a "Plastic" state. In other words, when all the steel has "Plastically Yielded" in compression and the entire concrete section is subjected to its maximum compressive stress. The plastic state is only valid when used with an Elasto-Plastic steel model. Therefore the effect of strain hardening and softening in the steel is ignored.

11. A perfect bond exists between concrete and steel to ensure equilibrium and compatibility of strains.
12. Tensile and compressive stress-strain relationship of steel reinforcement is identical.
13. Steel reinforcement is represented as a discrete point rather than a circular area. Therefore the stress in the reinforcing bar is computed based on the strain at the centroid of the rebar.
14. In the compression region, the concrete displaced by the actual area of the reinforcement is deducted from the Whitney stress block.

Sign Convention

In order for the user to understand the results of the program a sign convention must be specified and adhered to:

1. Based on the right hand rule, for a plan view of the cross-section the positive X-axis will point to the right, the positive Y-axis will point to the top, and the positive Z-axis will point up and out of the plane.
2. Positive net axial loads are compressive and negative net axial loads are tensile. Compressive net axial loads act in the negative Z direction.
3. With the axis and compressive and tensile axial directions defined the right hand rule applies as follows: A positive moment about the x-axis (M_x) will produce compression (+) at the bottom face and tension (-) at the top face of the cross section. A positive moment about the y-axis (M_y) will produce compression (+) at the right face and tension (-) at the left face of the cross-section.
4. Positive rotation is in the clockwise direction as shown in Figure 1.

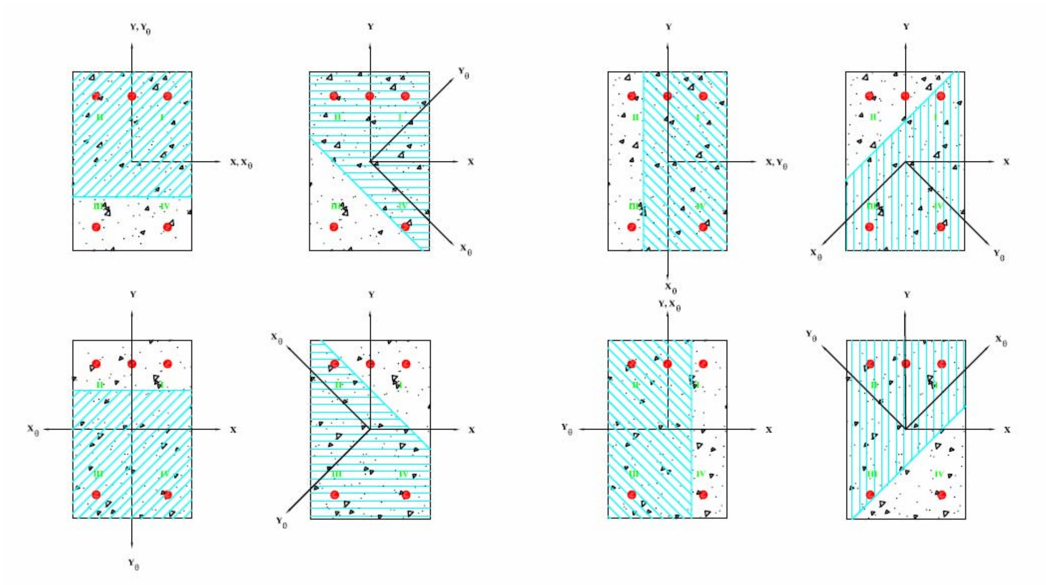

Figure 1. Cross-Section rotated in Clockwise Rotation at 45° intervals.

Therefore as the orientation of the neutral axis is rotated from 0° to 360° and the corresponding moments M_x and M_y will be as follows:

Quadrant I	0° - 90°	$-M_x$, $+M_y$
Quadrant IV	90° - 180°	$+M_x$, $+M_y$
Quadrant III	180° - 270°	$+M_x$, $-M_y$
Quadrant II	270° - 360°	$-M_x$, $-M_y$

Coordinate System

For building codes the origin O_{xy} of the coordinate system may be chosen as either the geometric centroid ($X_{G.C.}$, $Y_{G.C.}$) or the plastic centroid ($X_{P.C.}$, $Y_{P.C.}$) dependent on the user's preference. The plastic centroid coordinate system can be determined when the section is in a "Plastic" state. In other words, when all the steel has "Plastically Yielded" in compression (Elasto-Plastic Steel Model) and the entire concrete section is subjected to its maximum compressive stress (Wang & Salmon, 1998). A "Plastic" state can only be achieved in a theoretical elasto-plastic model since steel has stress hardening and softening effects at large strains.

The advantage in selecting the plastic-centroid for the origin of the coordinate system is that at the maximum axial compressive force, P_0, the corresponding moment will be zero.

The geometric centroid of the section is determined by computing the cross-sectional area of the polygon and its static moments about the X and Y axis initially prescribed by the user. The origin of the geometric centroid ($O_{G.C.}$) of the section is then determined by dividing the respective static moments by the cross-sectional area:

$$X_{G.C.} = e_{XG.C.} = S_{yc} / A_c$$
$$Y_{G.C.} = e_{yG.C.} = S_{xc} / A_c$$

If the contribution of the steel is added, the plastic-centroid of the section can be determined in a similar manner. The cross-sectional area (A_s) of the transformed reinforcement and each bar's respective static moment (Sx_s, Sy_s) can also be calculated. The origin of the plastic centroid ($O_{P.C.}$) of the section is computed by dividing the combined static moments of the transformed steel and concrete contributions by the total area of the two materials:

$$X_{P.C.} = e_{XP.C.} = [S_{yc} + S_{ys}] / [A_c + A_s]$$
$$Y_{P.C.} = e_{yP.C.} = [S_{xc} + S_{xs}] / [A_c + A_s]$$

The initial concrete and steel coordinates of the cross section (X_{oi}, Y_{oi}) can be transformed to the origin of the geometric centroid or plastic centroid by the following equations:

Geometric Centroid:

$$X_i = X_{oi} - X_{G.C.}$$
$$Y_i = Y_{oi} - Y_{G.C.}$$

Plastic Centroid:

$$X_i = X_{oi} - X_{P.C.}$$
$$Y_i = Y_{oi} - Y_{P.C.}$$

The transformed coordinate system enables the rotation of the section about the geometric or plastic centroid:

$$X_{\theta i} = X_i \cos\theta + Y_i \sin\theta$$
$$Y_{\theta i} = -X_i \sin\theta + Y_i \cos\theta$$

The X-axis will remain parallel to the plane in which the neutral axis will be incremented within for each angular rotation, - θ. The vertical location with respect to the centroid can thereby be determined for each neutral axis increment, c. The intersection of the neutral axis with the polygonal sides of the cross-section allows for the determination of the portions of the cross-section which lie within the compression or tension regions.

Moments of Inertia and Principal Axes

Green's theorem and the transformation of the double line integrals to a single line integral about the perimeter of the compression region can further be used to compute the second moments of inertia and the principal axes of the cross-section.

The user is often interested in the second moments of inertia about the primary axes (Ixx, Iyy), the product of inertia (I_{xy}), and the section's principal axes ($\theta_{1,2}$) and moments of inertia about them (I_1, I_2), respectively. The calculation of the concrete cross-section's second moments of inertia (Ixx_c, Iyy_c) and product of inertia (Ixy_c) about the geometric centroid is given by the following formulae:

$$Ixx_c = \int y^2 \, dA + A \, d_y^2$$
$$Iyy_c = \int x^2 \, dA + A \, d_x^2$$
$$Ixy_c = \int xy \, dA + A \, d_x \, d_y$$

The principal axes ($\theta_{1,2}$) and their corresponding moments of inertia about them (I_1, I_2) are calculated by the equations:

$$\theta_P = \tfrac{1}{2} \, TAN^{-1} \left[-2Ixy_c \,/\, (Ixx_c - Iyy_c)\right]$$

$$I_{1,2} = \tfrac{1}{2} \, (Ixx_c + Iyy_c) \pm \left[{}^{1}/_{4} \, (Ixx_c - Iyy_c)^2 + Ixy_c^{\,2}\right]^{1/2}$$

If the transformed sectional properties are of interest, the transformed steel contribution can be added. Generally the working stress method utilizes the modular ratio $\eta = E_s/E_c$ and the transformed steel area is computed as $A_s \, (\eta - 1)$. In the ultimate strength method the user seeks an equivalent (transformed) steel Area A_s that will give the same strength of the section. Here the ratio of (f_y / f_c' – 1) is used in lieu of (E_s/E_c – 1). The moment of inertia (Ixx_s, Iyy_s, Ixy_s) contribution from the transformed steel reinforcement is given by:

$$Ixx_s = \int A_s \, y^2 \, (f_y \,/\, f_c' - 1)$$
$$Iyy_s = \int A_s \, x^2 \, (f_y \,/\, f_c' - 1)$$
$$Ixy_s = \int A_s \, x \, y \, (f_y \,/\, f_c' - 1)$$

Therefore the transformed section's principal axes and corresponding moments of inertia are then:

$$\theta_{PT} = \tfrac{1}{2} \, TAN^{-1} \left[-2[Ixy_c + Ixy_s] / \, [(Ixx_c + Ixx_s) - (Iyy_c + Iyy_s)]\right]$$

$$I_{1,2T} = \tfrac{1}{2} \, [(Ixx_c + Ixx_s) + (Iyy_c + Iyy_s)] \pm \left[{}^{1}/_{4} \, [(Ixx_c + Ixx_s) - (Iyy_c + Iyy_s)]^2 + [Ixy_c + Ixy_s]^2\right]^{1/2}$$

Elasto-Plastic Steel Model

The ACI-318 uses the elasto-plastic stress-strain model. The steel reinforcement behaves elastically with a slope equal to the modulus of elasticity, E_s, up to a specified yield plateau. The yield strain, ε_y, is defined as:

$$\varepsilon_y = F_y \,/\, E_s$$

F_y is the stress value associated with the yield plateau of a typical ASTM tensile bar test. The yield strain is the transition point beyond which the model behaves perfectly plastic with a constant stress value of $f_s = f_y$. The elasto-plastic steel model used in the ACI-318 building code does not define a strain hardening or softening point or an ultimate strain of the reinforcement beyond which the bar breaks and the stress in the steel becomes zero, $f_s = 0$. These large strains at which the reinforcement could potentially fail are generally uncommon in column design but should be evaluated by the user when analyzing large concrete cross-sectional models.

Therefore, the elasto-plastic theoretical steel model has the following stress-strain relationships:

$$f_s = E_s\,\varepsilon_s \qquad \text{for } 0 \le \varepsilon_s \le \varepsilon_y$$
$$f_s = F_y \qquad \text{for } \varepsilon_s > \varepsilon_y$$

Figure 2 illustrates the elasto-plastic stress-strain relationship for Grade 60 steel. Note that no defined ultimate strain, ε_{su}, appears in this steel model.

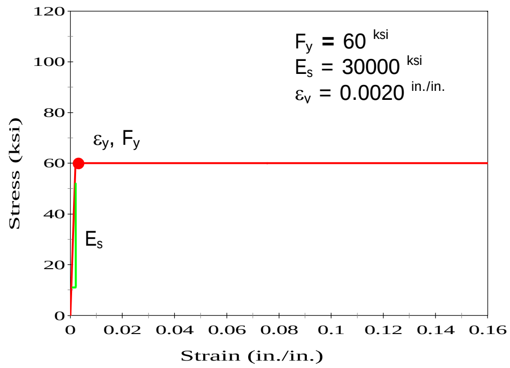

Figure 2. Elasto-Plastic Steel Model (F_y = 60 ksi, E_s = 30,000 ksi)

The user has the option of computing moments about the geometric centroid of the cross-section or about the plastic centroid of the cross-section.

ACI-318 Concrete Model

The ACI-318 concrete model is simplified with a rectangular stress block which can withstand a maximum compressive strain of ε_{cu} = 0.0030 in./in. Figure 3 depicts the relationship of the Whitney stress block with a linear strain relationship. At the maximum strain ε_{cu} = 0.0030 in./in. the maximum stress in the block is 0.85 f_c'. The depth of the stress block is defined as $a = \beta_1\,c$ where c is the depth of the neutral axis.

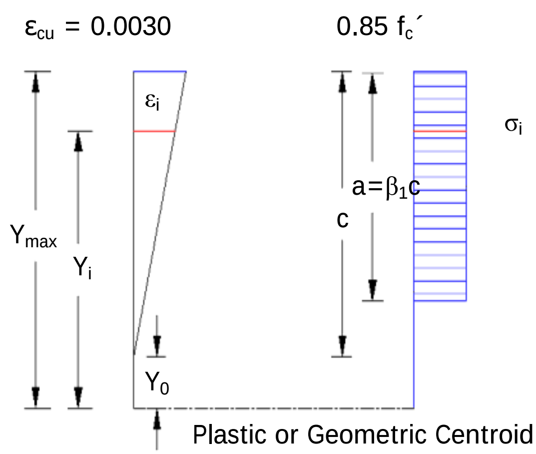

Figure 3. ACI-318 Concrete Model Stress-Strain Relationship

The definition of the "Y" variables is measured from the either the geometric centroid or plastic centroid to the strained fiber represented by the appropriate subscript:

Y_i = strained fiber at level "i"
Y_{max} = maximum compressive fiber
Y_0 = "zero" strain fiber in cross-section

The other variables shown in the figure are defined as:

ε_{cu} = ultimate concrete strain in maximum fiber
β_1 = Whitney block reduction coefficient for ACI-318
f_c' = Concrete 28 day ultimate design compressive strength
ε_i, σ_i = concrete stain and equivalent stress, respectively, at distance Y_i
c = depth of neutral axis
a = effective compressive block depth for ACI-318 concrete model

The "Whitney" coefficient, β_1, is defined as:

$\beta_1 = 0.85$ $f_c' \leq$	4 ksi
$\beta_1 = 1.05 - 0.05\ f_c'$	4 ksi $< f_c' \leq$ 8 ksi
$\beta_1 = 0.65$ $f_c' >$	8 ksi
$\beta_1 = 0.85$ $f_c' \leq$	30 MPa
$\beta_1 = 1.0643 - 0.007143\ f_c'$	30 MPa $< f_c' \leq$ 58 MPa
$\beta_1 = 0.65$ $f_c' >$	58 MPa

Figure 4 is a graphical representation of the reduction coefficient, β_1, versus concrete strength, f_c'. The Whitney coefficient simply reduces the effective area over which the equivalent concrete compressive stress block acts.

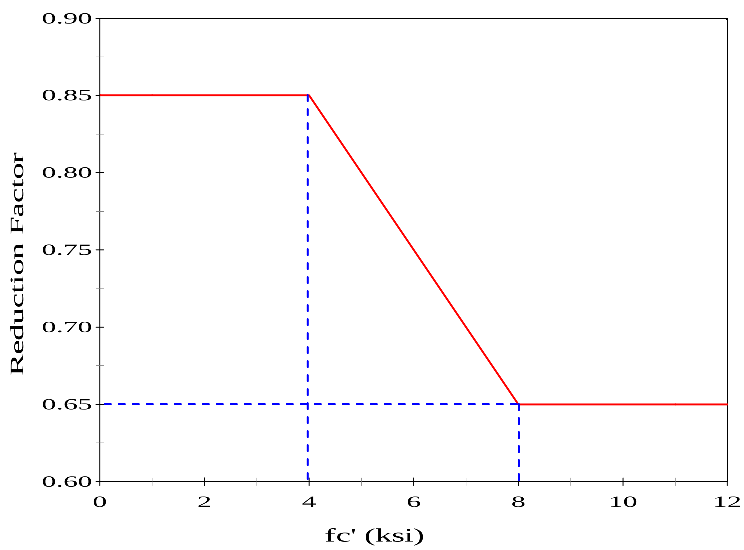

Figure 4. Whitney Block Reduction Coefficient, β_1.

Incorporating these concrete model variables into a rectangular cross section example the stress-strain relationship and depiction of the variables appears below in Figure 5.

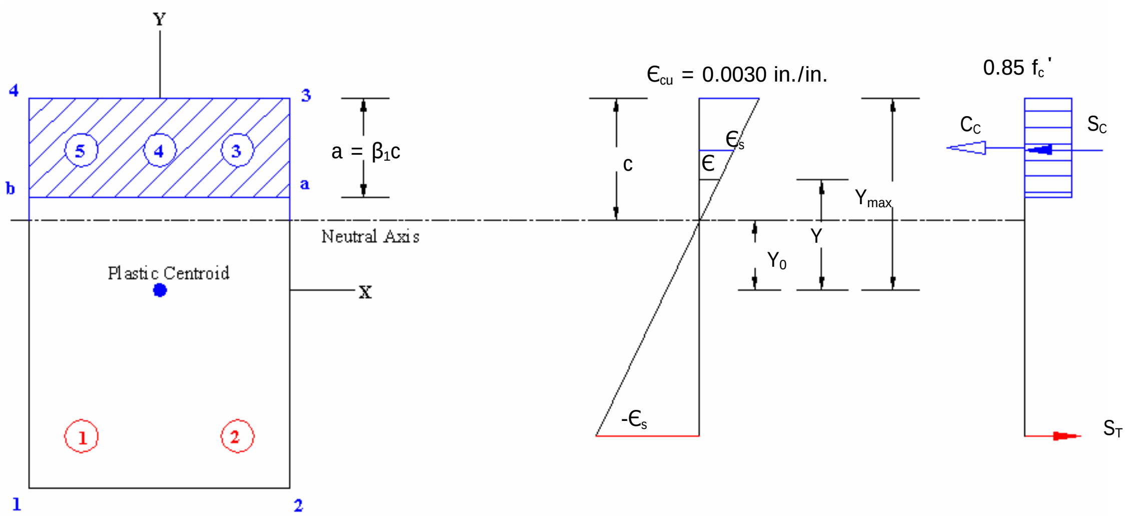

Figure 5. ACI-318 Stress-Strain Relationship and Graphical Depiction of Variables

Figures 6 through 8 illustrate the various common visual depictions of a rectangular reinforced concrete cross-section's axial versus moment capacity diagrams. Figure 6 shows the 3-dimensional surface of the theoretical relationship between the axial load and the bi-axial moments for a single quadrant of the diagram.

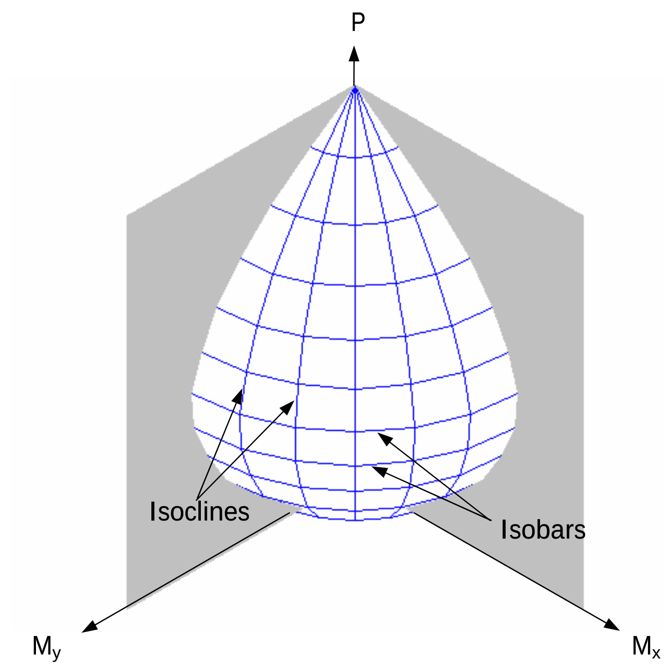

Figure 6. 3-D Surface of Axial load versus Bending Moment Capacity.

The isocline diagram shown in Figure 7 depicts the nominal axial load versus moment capacity at an arbitrarily selected angular rotation.

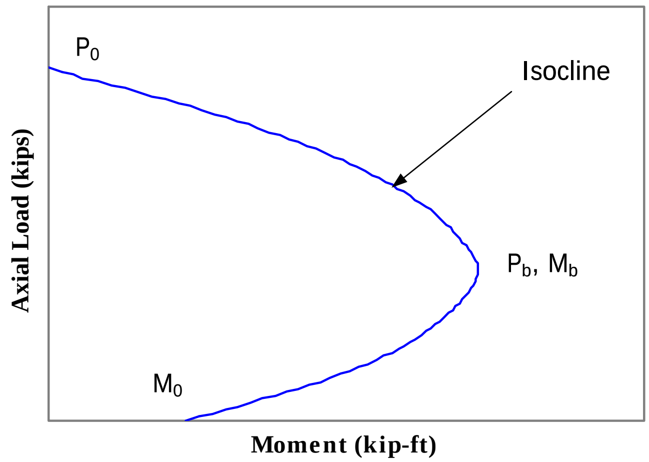

Figure 7. Isocline – Axial Load versus Bending Moment.

The isobar diagram shown in Figure 8 depicts the nominal moment M_x versus M_y for an arbitrarily selected axial load increment.

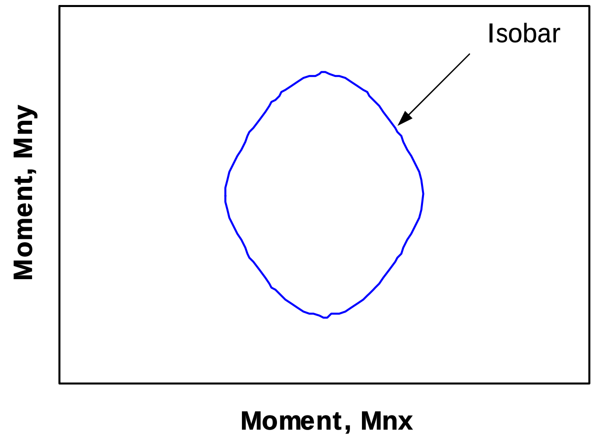

Figure 8. Isobar – Bending Moments at Axial Load Increment.

ACI-318 – Rectangular Cross-Section Example

θ = 45°, c = 27.46 in., f_c' = 5,000 psi, β_1 = 0.80, ε_{cu} = 0.0030 in./in., Y_{max} = 17.515 in., Y_0 = -9.945 in

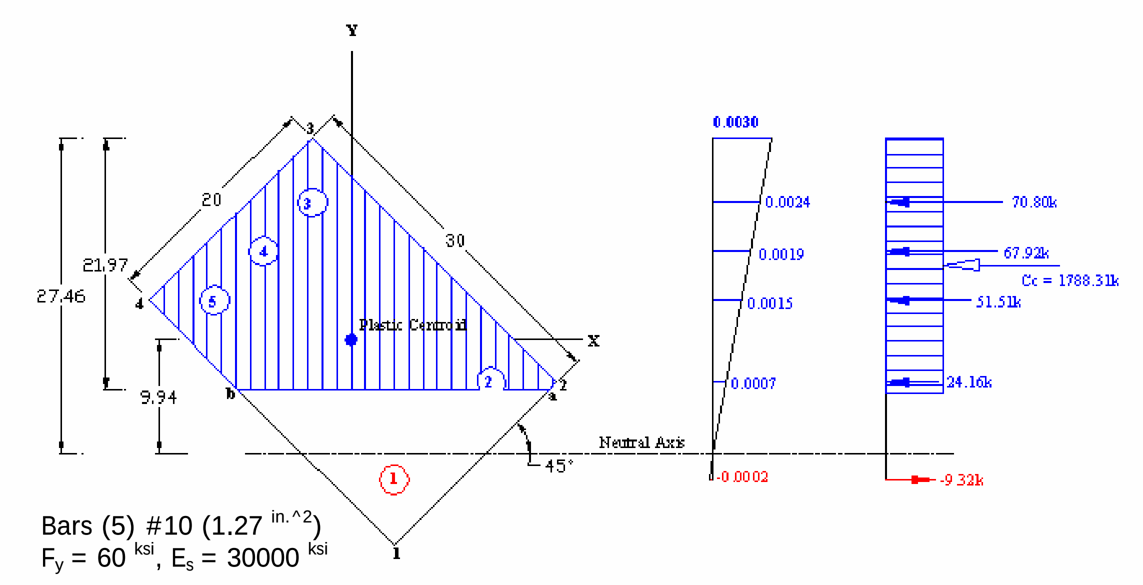

Figure 9. Rectangular Reinforced Concrete Section Example

The concrete cross sectional coordinates and the cross-sectional properties about the geometric centroid can be computed as follows:

Table 1. Concrete Coordinates and Cross-Sectional Properties

	X (in.)	Y (in.)	A_c (in.^{2})	Sx_c (in.^{3})	Sy_c (in.^{3})	I_{xx} (in.^{4})	I_{yy} (in.^{4})	I_{xy} (in.^{4})
1	0	0	0	0	0	0	0	0
2	20	0	0	0	0	0	0	0
3	20	30	600	9000	6000	180000	80000	90000
4	0	30	0	0	0	0	0	0
			600	**9000**	**6000**	**180000**	**80000**	**90000**

Similarly the steel reinforcement transformed cross-sectional properties can be computed as follows:

$$A_s = A_{sj}\,(f_y/f_c' - 1)$$
$$S_{xs} = A_{sj}\,y_{sj}\,(f_y/f_c' - 1)$$
$$S_{ys} = A_{sj}\,x_{sj}\,(f_y/f_c' - 1)$$

Table 2. Steel Coordinates and Transformed Cross-Sectional Properties

	X (in.)	Y (in.)	Bar #	A_s (in.^{2})	F_y (ksi)	E_s (ksi)	ϵ_y (in./in.)	Transformed Cross-Sectional Properties		
								A_s (in.^{2})	Sx_s (in.^{3})	Sy_s (in.^{3})
1	4	4	10	1.27	60	30000	0.0020	13.97	55.88	55.88
2	16	4	10	1.27	60	30000	0.0020	13.97	55.88	223.52
3	16	26	10	1.27	60	30000	0.0020	13.97	363.22	223.52
4	10	26	10	1.27	60	30000	0.0020	13.97	363.22	139.70
5	4	26	10	1.27	60	30000	0.0020	13.97	363.22	55.88
								69.85	**1201.42**	**698.50**

Therefore the plastic centroid location can be determined by the following equations:

$$X_{P.C.} = e_{XP.C.} = [S_{yc} + S_{ys}] / [A_c + A_s] = 10.000 \text{ in.}$$
$$Y_{P.C.} = e_{yP.C.} = [S_{xc} + S_{xs}] / [A_c + A_s] = 15.229 \text{ in.}$$

The section's origin can now be located at the plastic centroid and the cross-section's concrete and steel new coordinates relative to the plastic centroid can not be computed by the following equations:

$$X_i = X_{oi} - X_{P.C.}$$
$$Y_i = Y_{oi} - Y_{P.C.}$$

Table 3. Concrete Coordinate Transformation to Plastic Centroid Origin

	$X_{P.C.}$ (in.)	$Y_{P.C.}$ (in.)
1	-10	-15.229
2	10	-15.229
3	10	14.771
4	-10	14.771

Table 4. Steel Coordinate Transformation to Plastic Centroid Origin

	$X_{P.C.}$ (in.)	$Y_{P.C.}$ (in.)
1	-6	-11.229
2	6	-11.229
3	6	10.771
4	0	10.771
5	-6	10.771

The transformed coordinate system enables the rotation of the orientation of the neutral axis about the plastic centroid origin by the following equations:

$$X_{\theta i} = X_i \cos\theta + Y_i \sin\theta$$
$$Y_{\theta i} = -X_i \sin\theta + Y_i \cos\theta$$

Table 5. Concrete Coordinate Transformation to Angular Rotation

	X_θ (in.)	Y_θ (in.)
1	3.698	-17.840
2	17.840	-3.698
3	-3.373	17.515
4	-17.515	3.373

Table 6. Steel Coordinate Transformation to Angular Rotation

	X_θ (in.)	Y_θ (in.)
1	3.698	-12.183
2	12.183	-3.698
3	-3.373	11.859
4	-7.616	7.616
5	-11.859	3.373

The depth of the compression region ($a = \beta_1 c$ = 0.80 x 27.46 in = 21.97 in.) creates the coordinates a = (17.085, -4.453) and b = (-9.690,-4.453) along the sides of the cross-section. The ACI-318 stress block coefficients are A = 1, B = 0, and C = 0 and the contributions of each side of the concrete cross-section within the compression region can be computed as follows:

Table 7. Concrete Coordinates in Compression, J_i, A_0, B_0, C_c, Mx_c, and My_c

	X_θ (in.)	Y_θ (in.)	J_0	J_1	J_2	J_3	J_4	A_0	B_0	C_c (kips)	Mx_c (in.-kip)	My_c (in.-kip)
Rectangular Compression Region (a-2-3-4-b)												
a	17.085	-4.453										
2	17.840	-3.698	0.75	-3.08	12.57	-51.52	211.74	21.54	1.00	56.02	228.13	489.18
3	-3.373	17.515	21.21	146.56	1808.05	23483.45	329851.82	14.14	-1.00	652.12	-1124.58	4048.93
4	-17.515	3.373	-14.14	-147.71	-1778.40	-23497.82	-329626.19	-20.89	1.00	627.75	-5554.73	-3779.10
b	-9.690	-4.453	-7.83	4.22	-42.22	65.89	-437.36	-14.14	-1.00	452.42	74.38	-3161.89
										1788.31	**-6376.80**	**-2402.88**

Similarly the location of each reinforcing bar relative to the neutral axis can be determined along with the tensile or compressive strain and stress in each bar. The contribution of each bar would thereby be as follows:

Table 8. Steel Coordinates, Strain, Stress, Force, Mx_s, and My_s

	X_θ (in.)	Y_θ (in.)	Bar #	A_s (in.^{2})	E_s (ksi)	f_y (ksi)	ϵ_y (in./in.)	ϵ_S (in./in.)	σ_S (ksi)	$\Delta f_c'$ (ksi)	Force (kips)	Mx_s (in.-kip)	My_s (in.-kip)
1	3.698	-12.183	10	1.27	30000	60	0.0020	-0.0002	-7.34		-9.32	-113.52	-34.45
2	12.183	-3.698	10	1.27	30000	60	0.0020	0.0007	20.47	-4.25	24.16	89.34	294.34
3	-3.373	11.859	10	1.27	30000	60	0.0020	0.0024	60.00	-4.25	70.80	-839.62	-238.84
4	-7.616	7.616	10	1.27	30000	60	0.0020	0.0019	57.55	-4.25	67.92	-517.25	-517.25
5	-11.859	3.373	10	1.27	30000	60	0.0020	0.0015	43.65	-4.25	51.51	-173.75	-610.81
											205.07	**-1554.80**	**-1107.01**

Transforming the moments to the original coordinate system and applying the ACI-318-02 reduction factors (ϕ = 0.65, θ = 1.0) to the triplet yields:

$P_n = P_c + P_s$
$= 1788.31 + 205.07$
$= +1993.38$ kips
$\phi\,\theta\,P_n = +1295.70$ kips (Compression)

$M_{nx} = (M_{cx} + M_{sx}) \cos(-\theta) + (M_{cy} + M_{sy}) \sin(-\theta)$
$= (-6376.80 + -1554.80) \cos(-45^\circ) + (-2402.88 + -1107.01) \sin(-45^\circ) = -8090.35$ in.-kip
$= -674.20$ ft-kips
$\phi M_{nx} = -438.23$ ft-kips

$M_{ny} = -(M_{cx} + M_{sx}) \sin(-\theta) + (M_{cy} + M_{sy}) \cos(-\theta)$
$= -(-6376.80 + -1554.80) \sin(-45^\circ) + (-2402.88 + -1107.01) \cos(-45^\circ) = 3126.61$ in.-kip
$= 260.55$ ft-kips
$\phi M_{ny} = 169.36$ ft-kips